UNIT 2 : REAL NUMBER SYSTEM-II

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2.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- neighbourhood of a point on the real line.
- limit point of a set, interior of a set.
- bolzano-Weierstrass Theorem
- open sets, closed sets, closure of a set
- countability of sets.

2.2 INTRODUCTION

In this unit we will discuss the points and subsets of the real line such as neighbourhood of a point, open sets, closed sets, limit points of a set, etc. These are known as the topological properties of the real line.

2.3 NEIGHBOURHOOD OF A POINT

Let p be a real number. A subset S of $\underline{\underline{\mathrm{R}}}$ is said to be a neighbourhood of p if there exists an open interval]a, b[such that

$$p \in]a,\ b[\subseteq \mathrm{S}$$

This is equivalent to saying that there exists an open interval of the form $]p-\delta,\ p+\delta[$ for same $\delta > 0$ such that

$$]p-\delta, p+\delta[\subseteq s$$

$]p-\delta, p+\delta[$ is called the δ-neighbourhood of p.

For convenience, we will write nbd for neighbourhood.

Note: 1. An open bounded interval containing the point p is a neighbourhood of p.

2. A closed bounded interval containing the point p may not be a neighbourhood of p.

Let us consider the closed interval [0, 1]. [0, 1] is not a nbd of 9, for there does not exist any open interval containing 0 which is a subset of [0, 1]. Similarly, [0, 1] is not a nbd of 1.

A closed bounded interval [*a, b*] is a nbd of each of the points of] *a, b* [, but not of the end points *a* and *b*.

3. The null set ϕ is a nbd of each of its point in the sense that there is no point in ϕ of which it is not a nbd.

4. The set of real numbers R is a nbd of each real number p, for any open interval containing p is a subset of R.

5. The set of rational numbers Q is not a nbd of any of its points p, for any open interval containing p will contain infinite number of irrotional numbers. Clearly the open interval cannot be a subset of Q.

6. The singleton set {p} for any $\mathrm{p} \in \mathrm{R}$ is not a nbd of p, for there does not exist any open interval containing p which is a subset of {p}.

7. A nonempty finite subset S of R is not a nbd of any point.

It S is a nbd of p, then there exists an open interval containing p which is subset of S. Now the open interval contains an infinite number of points, whereas S is finite. Thus, S cannot be a nbd of p.

Theorem 2.1 The union of any two nbds of a point is a nbd of the point.

Proof : Let S_1, S_2 be two subsets of R.

Let $p \in R$

Let S_1, S_2 be two nbds of p.

$\because$ S_1 is a nbd of p, there exists open interval [a_1, b_1 [such that

$p \in] a_1, b_1 [\subseteq S_1$

Again, $\because$ S_2 is a nbd of p, there exists open interval [a_2, b_2 [such that

$p \in] a_2, b_2 [\subseteq S_2$

Now $p \in] a_1, b_1 [\Rightarrow a_1 < p < b_1$

$p \in] a_2, b_2 [\Rightarrow a_2 < p < b_2$

Now $a_1 < b_1, a_1 < b_2, a_2 < b_1, a_2 < b_2$

Let a_3 = min $\{a_1, a_2\}$

b_3 = max $\{b_1, b_2\}$

Then $]a_1, b_1[\cup]a_2, b_2[=]a_3, b_3[$

Also $p \in]a_3, b_3[$

Now $]a_1, b_1[\leq S_1 \cup S_2$, $]a_2, b_2[\subseteq S_1 \cup S_2$

$\therefore]a_3, b_3[=]a_1, b_1[\cup]a_2, b_2[\subseteq S_1 \cup S_2$

Thus $p \in]a_3, b_3[\subseteq S_1 \cup S_2$

$\therefore$ $S_1 \cup S_2$ is a nbd of p.

Note: The union of a finite number of nbds of p is a nbd of p.

Theorem 2.2 The intersection of any two nbds of a point is a nbd of the point.

Proof : Let S_1, S_2 be two subsets of R.

Let $p \in R$

Let S_1, S_2 be two nbds of p.

$\because$ S_1 is a nbd of p, there exists open interval] a_1, b_1 [such that

$p \in]a_1, b_1[\subseteq S_1$

Again, $\because$ S_2 is a nbd of p, there exists open interval] a_2, b_2 [such that

$p \in]a_2, b_2[\subseteq S_2$

Now $a_1 < b_1, a_1 < b_2, a_2 < b_1, a_2 < b_2$

Let a_3 = max $\{a_1, a_2\}$

b_3 = min $\{b_1, b_2\}$

Then $]a_1, b_1[\cap]a_2, b_2[=]a_3, b_3[$

Also $p \in]a_3, b_3[$

Now $]a_3, b_3[=]a_1, b_1[\cap]a_2, b_2[\subseteq]a_1, b_1[\leq S_1$

$]a_3, b_3[=]a_1, b_1[\cap]a_2, b_2[\subseteq]a_2, b_2[\subseteq S_2$

$\therefore]a_3, b_3[\leq S_1 \cap S_2$

This $p \in]a_3, b_3[\leq S_1 \cap S_2$

$\therefore S_1 \cap S_2$ is a nbd of p.

Note 1. The intersection of a finite number of nbds of p is a nbd of p.

2. The intersection of an infinite number of nbds of p may not be a nbd of p.

Let us consider the family of open intervals $]-1,1[, \left]-\frac{1}{2}, \frac{1}{2}\right[, \left]-\frac{1}{3}, \frac{1}{3}\right[, \left]-\frac{1}{4}, \frac{1}{4}\right[, \ldots, \left]-\frac{1}{n}, \frac{1}{n}\right[, \ldots$

Each of these open intervals is a nbd of 0.

Now $]-1,1[\cap \left]-\frac{1}{2}, \frac{1}{2}\right[\cap \left]-\frac{1}{3}, \frac{1}{3}\right[\cap \ldots \cap \left]-\frac{1}{n}, \frac{1}{n}\right[\cap = \bigcap_{n=1}^{\infty} \left]-\frac{1}{n}, \frac{1}{n}\right[= \{0\}$

Clearly {0} is not a nbd of 0.



CHECK YOUR PROGRESS-1

Q.1: If S is a nbd of a point p, and $S \subseteq T$, show that T is also a nbd of p.

Q.2: Can a set S be a nbd of any point of R – S? Justify your answer.

Q.3: Give an example of each of the following :

(a) a set which is a nbd of each of its points.

(b) a set which is not ao nbd of any of its points.

Do You Know ?

Let p be any point on the real line. Let $\delta > 0$ be a real number. A set consisting of all those points on the real line which are at a distance δ from p is called a neighbourhood (nbd) of p. Thus, the set $\{x : x \in R, |x - p| < \delta$

i.e., $\{x : p - \delta < x < p + \delta\}$

i.e., $]p - \delta, p + \delta[$

is a nbd of p.

2.4 LIMIT POINT OF A SET

Let S be a subset of R. A real number p is said to be a limit point (or a cluster point, or an accumulation point) of S if every nbd of p contains a point of S other than p.

The null set has no limit point.

Note : The set of all limit points of S is called the derived set of S. Generally it is denoted by S'.

Examples

1. Let us consider the set S = [0, 1 [

Clearly $0 \in S$

Let us consider the nbd] – 0.5, 0.5 [of 0.

We have 0.1 $\in$] – 0.5, 0.5 [

Also 0.1 $\in$ S.

Thus we can find the point 0.1 $\in$ S. Also we note that $0.1 \neq 0,$ i.e, the nbd] – 0.5, 0.5 [of 0 contains point of S other than 0.

In this way, we will see that every nbd of 0 contains point of S different from 0.

$\therefore$ 0 is a limit point of S.

We note that the limit point 0 belongs to S.

2. Let us consider the set

$$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots\ldots\right\}$$

$$= \left\{\frac{1}{n} : n \in N\right\}$$

Here 0 does not belong to S.

Let us consider the nbd] – 0.5, 0.5 [of 0.

We have 0.25 6] – 0.5, 0.5 [

Also $0.25 = \frac{1}{4} \in S$

Thus we can find 0.25 $\in$ S.

Also we note that $0.25 \neq 0,$ i.e. the nbd] – 0.5, 0.5 [of 0 contaions point of S other than 0.

In this way, we will see that every nbd of 0 contains point of S different from 0.

$\therefore$ 0 is a limit point of 0.

Also we note that 0 is the only limit point of S.

We note that the limit point 0 does not belong to S.

3. Let us consider the set N of natural numbers.

For every real number p, we can always find $\delta > 0$ such that $]p-\delta,\ p+\delta[$ does not contain a point of N different from p.

Thus, N has no limit point.

4. Let us consider the set Q of rational numbers.

We know that between any two rational numbers, there exist infinite number of rational numbers. Also there exist infinite number of rationals between any two irrational numbers.

Thus, every point of R is a limit point of Q.

From the above examples, we note the following :

FACTS

1. A limit point of a set may or may not belong to the set.
2. A set may have no limit point.
3. A set may have only one limit point.
4. A set may have more than one limit point.

Theorem 2.3 : Let A, B be two subsets of R. Then

(a) $A \subseteq B \Rightarrow A' \subseteq B'$

(b) $(A \cup B)' = A' \cup B'$

(c) $(A \cap B)' = A' \cap B'$

Proof (a) Let $p \in A'$

This $\Rightarrow$ p is a limit point of A.

$\therefore$ For each $\delta > 0$, the nbd $]p-\delta,\ p+\delta[$ contains a point, say p_1, of A other than p.

Now $p_1 \in A \Rightarrow p_1 \in B, \because A \subseteq B$

Thus the nbd $]p-\delta,\ p+\delta[$ contains a point $p_1 \in B$ other than p.

$\therefore$ p is a limit point of B, i.e, $p \in B'$

Thus $p \in A' \Rightarrow p \in B'$

$\therefore A' \subseteq B'$

(b) We know that $A \subseteq A \cup B$

$\therefore A' \subseteq (A \cup B)'$, by (a)

Again, $B \subseteq A \cup B$

$\therefore B' \subseteq (A \cup B)'$, by (a)

$\therefore A' \cup B' \subseteq (A \cup B)'$ (1)

Let $\therefore p \in (A \cup B)'$

$\therefore$ p is a limit point of $A \cup B$.

$\therefore$ Every nbd of p contains a point $q \in A \cup B,\ q \neq p$

$\therefore$ Every nbd of p contains a point ($q \in A$ or $q \in B$), $q \neq p$

$\therefore$ Every nbd of p contains a point $q \in A,\ q \neq p$

Or every nbd of p contains a point $q \in B,\ q \neq p$

$\therefore$ p is a limit point of A or p is a limit point of B.

i.e., $p \in A'$ or $p \in B'$

i.e., $p \in A' \cup B'$

$\therefore (A \cup B)' \subseteq A' \cup B'$ (2)

From (1) and (2), we get

$(A \cup B)' = A' \cup B'$

(c) We know that $A \cap B \subseteq A$

$\therefore\ (A \cap B)' \subseteq A'$, by (a)

Again $A \cap B \subseteq B$

$\therefore\ (A \cap B)' \subseteq B'$, by (b)

$\therefore\ (A \cap B)' \subseteq A' \cap B'$

Do You Know ?

1. Every point of [*a, b*] is its limit point. A point not belonging to [*a, b*] is not a limit point. Thus $[a,b]' = [a,b]$
2. Every point of] *a, b* [is its limit point. Also the points, *a, b* are its limit point. Thus,

$$]a,b[' = [a, b].$$

Note: $(A \cap B)' \neq A' \cap B'$

Let us consider $A =]2, 3[,\ B =]3, 4['$

$A' =]2, 3[' = [2, 3]$

$B' =]3, 4[' = [3, 4]$

$\therefore A' \cap B = \{3\}$

Again $A \cap B = \phi$

$$(A \cap B)' = \phi' = \phi$$

$$\therefore (A \cap B)' \neq A' \cap B'.$$

Theorem 2.4 : Let S be a subset of $\underline{\underline{R}}$. Let p be a limit point of S. Then every nbd of p contains an infinite number of elements of S.

Proof : Let p be a limit point of S. Then every nbd of p contains at least one point of S different from p. Let $]p - \delta_1, p + \delta_1[$ be a nbd which contains a point $p_1 \in S, p_1 \neq p$

Let $|p_1 - p| = \delta_2 < \delta_1$

Now, let us consider the nbd $]p - \delta_2, p + \delta_2[$ of p.

It must contain a $p_2 \in S, p_2 \neq p$.

Continuing the above process for the nbd $]p - \delta_3, p + \delta_3[$ of p, where $\delta_3 = |p_2 - p|$, and so on, we conclude that the nbd $]p - \delta_1, p + \delta_1[$ contains an infinite number of elements of S.

Theorem 2.5 : A finite set has no limit points.

Proof: Let S be a finite set.

Let S = $\{x_1, x_2, ..., x_m\}$, where all the x_1's are distinct.

Let p be a real number.

Now p cannot be a limit point of S, for if p is a limit point of S, then every nbd of p must contain an infinite number of elements of S, which is not possible, since S is a finite set.

$\therefore$ S has no limit point.

Do You Know ?

Let S be a subset of R. A real number p is a limit point of S if every nbd of p contains an infinite number of elements of S.

2.5 BOLZANO-WEIERSTRASS THEOREM

We have seen that a finite set has no limit point. An infinite set may or may not have limit points. For example, N has no limit pointl whereas every point of Q is a limit point of Q. Now one pertinent question arises. Under what conditions, an infinite set will have a limit point? (A finite set has no limit point). The problem was first studied by the Bohemian mathematician Bernard Bolzano (5 October 1781 - 18 December 1848), and later proved

by the German mathematician Karl Theodor Wilhelm Weierstrass (31 October 1815 - 19 February 1897). We are to going to prove this theorem, known as the Bolzano-Weierstrass Theorem.

Theorem 2.6 (Bolzano-Weierstrass)

Every infinite bounded subset of R has a limit point of R.

Proof : Let A be an infinite, bounded subset of R.

Since A is bounded, it must have both lower bound and upper bound.

Let m be a lower bound, and x be an upper bound of A.

Clearly $m \leq x \leq u, \forall x \in \mathrm{A}$.

Let S be a subset of R defined by $\mathrm{S} = \{x \in \underline{\underline{\mathrm{R}}} : x$ exceeds at most finite number of elements of A}

Thus m $\in$ S, M $\notin$ S

Clearly $\mathrm{S} \neq \phi$.

Since S is nonempty and bounded above, therefore, by completeness Axiom it must have a supremum in R.

Let p be the supremum of S.

$\therefore$ There is at least one element Z $\in$ S such that $\mathrm{Z} > \mathrm{p} - \delta$ for same $\delta > 0$.

Now Z $\in$ S. Therefore, Z exceeds at the most a finite number of elements of A. Thus the number of elements of A lying on the left of $\mathrm{p} - \delta$ is finite at the most, whereas the number of elements of A lying on the right of $\mathrm{p} - \delta$ is infinite.

Since p is the supremum of S, $\mathrm{p} + \delta \notin \mathrm{S}$. Thus, $\mathrm{p} + \delta$ exceeds an infinite number of elements of A. This implies that the number of elements of A lying on the left of $\mathrm{p} + \delta$ is infinite.

Thus, the number of elements on the right of $\mathrm{p} - \delta$ is infinite, and the number of elements on the left of $\mathrm{p} + \delta$ is infinite.

Hence, $]\mathrm{p} - \delta, p + \delta[$ contains an infinite number of elements of A.

$\therefore$ P is a limit point of A.

2.6 INTERIOR POINT, INTERIOR

We recall that a subset S $\subseteq$ R is said to be a nbd of a real number p if there exists an open interval] *a, b* [such that $p \in]a, b[\subseteq$ S. A nbd of p is also denoted by N(p).

Now, we define an interior point of a subset S of R.

Definition : Let S be a subset of R. A point $p \in S$ is said to be an interior point of S if there exists a nbd N (p) of p such that N (p) $\in$ S.

The set of all interior points of S is called the **interior** of S. Generally it is denoted by int S.

Note: The null set ϕ has no interior point. int ϕ = ϕ.

Example : Let us consider

$$S = \left\{ \frac{1}{n} : n \in N \right\}$$

$$= \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \right\}$$

Let $p \in S$

Clearly, every nbd of p contains points not belonging to S. So p cannot be an interior point of S.

$\therefore$ int S = ϕ

Explanation : Let $p = \frac{1}{2}, \delta = 0.1.$ The nbd $]\frac{1}{2} - 0.1, \frac{1}{2} + 0.1[$ of $\frac{1}{2}$ is not a subset of S, as it contains points not belonging to S.

2.7 OPEN SET

Let S be a subset of R. S is said to be an open set if every point of S is an interior point of S.

Note: For the null set ϕ, the condition required to be an open set is vacuously satisfied. ϕ is an open set.

Example : Let us consider

$$S = \left\{ \frac{1}{n} : n \in N \right\}$$

$$= \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \right\}$$

We know that no point of S is an interior point of S. Therefore, S is not an open set.

Do You Know ?

"In mathematics and logic, a vacuous truth is a conditional or universal statement (a universal statement that can be converted to a conditional statement) that is true because the antecedent cannot be satisfied." — Wikipedia.



CHECK YOUR PROGRESS- 2

Q.1: What is the interior of N? Is N an open set?

Q.2: What is the interior of Z? Is Z an open set?

Q.3: What is the interior of Q? Is Q an open set?

Q.4: What is the interior of R? Is R an open set?

Q.5: Let *a, b* be real numbers, and $a < b$. Example if (i)] *a, b* [, (ii) [*a, b* [, (iii)] *a, b*], (iv) [*a, b*] are open.

Q.6: Show that a nonempty finite set is not open.

Do You Know ?

1.] *a, b* [is an open set
2. [*a, b*] is not open

a, b are real numbers, $a < b$

Theorem 2.7 : Let S be a subset of R. Then S is an open set if and only if S = int S.

Proof : For the empty set ϕ, int $\phi = \phi$. Also ϕ is an open set. We assume that $S \neq \phi$.

Let S be open

Let $p \in S$

Then p is an interior point of S, i.e, $p \in$ int S

Then, $p \in S \Rightarrow p \in$ int S

$\therefore S \subseteq \text{int } S.$ (1)

Let q be an interior point of S. Then, by definition, $q \in S$

Thus, $q \in$ int $S \Rightarrow q \in S$.

$\therefore$ S = int S

Conversely, Let S = int S.

Let $p \in S$

$\Rightarrow p \in$ int S ($\because$ S = int S)

Thus, every point of S is an interior point of S.

$\therefore$ S is open.

Theorem 2. 8 : The union of two open sets in R is open.

Proof : Let G_1, G_2 be two open sets in R.

Let $p \in G_1 \cup G_2$

$\Rightarrow p \in G_1$ or $p \in G_2$

Let $p \in G_1$.

$\because$ G_1 is open, $\because$ p is an interior point of G_1.

$\therefore$ There exists a nbd N(p) of p such that $N(p) \subseteq G_1$.

Thus, $N(p) \subseteq G_i \subseteq G_1 \cup G_2$

$\therefore$ p is an interior point of $G_1 \cup G_2$

$\therefore$ $G_1 \cup G_2$ is an open set.

Similarly, if $p \in G_2$, it can be proved that $G_1 \cup G_2$ is an open set in R.

Theorem 2. 9 : The union of a finite number of open sets in R is open.

Proof : Let G_1, G_2, G_3, ..., G_n be n open sets in R.

Let $G = G_1 \cup G_2 \cup G_3 \cup .... \cup G_n$

Let $p \in G = G_1 \cup G_2 \cup G_3 ..... \cup G_n$

$\Rightarrow$ p belongs to at least one of the sets, say, G_1.

Now $p \in G_1$, and G_1 is open.

$\therefore$ p is an interior point of G_1

$\therefore$ There exists a nbd N(p) of p such that $N(p) \subseteq G_1$.

Thus, $N(p) \subseteq Gi \subseteq G = G_1 \cup G_2 \cup G_3 .... \cup G_n$

$\therefore$ p is an interior point of F.

$\therefore$ G is an open set.

Do You Know ?

An indexed class of sets, denoted by $\{G_i : i \in I\}$ assigns a set G_i to each $i \in I$. The set I is called the index set, the sets G_i are called indexed sets, and each $i \in I$ is called an index.

Theorem 2.10 : The union of an arbitrary collection of open sets in $\underline{\underline{R}}$ is open.

Proof: Let $\{G_i : i \in I\}$ be an arbitrary collection of open sets.

Let $G = \cup_{i \in I} G_i$

Let $p \in G$.

Then p belongs to at least one open set. Let it be G_j, $j \in I$

Now $p \in G_j$, and G_j is open.

$\therefore$ p is an interior of G_j

$\therefore$ There exists a nbd N(p) of p such that $N(p) \leq G_j$

Thus, $N(p) \leq G_j \leq G = \cup_{i \in I} G_i$

$\therefore$ p is an interoor of G.

$\therefore$ G is an open set.

Theorem 2.11 : The intersection of two open sets in R is open.

Proof : Let G_1, G_2 be two open sets in R. If $G_1 \cap G_2 = \phi$, there is nothing to prove. ϕ is an open set.

We assume that $G_1 \cap G_2 \neq \phi$

Let p $\in G_1 \cap G_2$

$\Rightarrow p \in G_1$ and $p \in G_2$

Now, $p \in G_1$, and G_1 is open.

$\therefore$ p is an interior point of G_1.

$\therefore$ There exists $\delta_1 > 0$ such that

$] p - \delta_1, p + \delta_1 [\subseteq G_1$

Again $p \in G_2$, and G_2 is open.

$\therefore$ p is an interior point of G_2.

$\therefore$ There exists $\delta_2 > 0$ such that

$] p - \delta_2, p + \delta_2 [\subseteq G_2$

Let δ = min $\{\delta_1, \delta_2\}$

Then $\delta > 0$

Now $] p - \delta, p + \delta [\not\subset] p - \delta_1, p + \delta_1 [\subseteq G_1$

$] p - \delta, p + \delta [\subseteq] p - \delta_2, p + \delta_2 [\subseteq G_2$

$\therefore$ $] p - \delta, p + \delta [\subseteq G_1 \cap G_2$

$\therefore$ p is an interior point of $G_1 \cap G_2$

$\therefore$ $G_1 \cap G_2$ is an open set.

Theorem 2.12: The intersection of a finite number of open sets in R is open.

Proof : Let G_1, G_2, ..., G_n be n open sets in R.

Let $G = G_1 \cap G_2 \cap G_3 \cap .... \cap G_n$

If G = ϕ, there is nothing to prove. ϕ is an open set.

We assume that $G \neq \phi$

Let $p \in G$

$\Rightarrow$ p $\in$ G_1, for each i = 1, 2, 3, ..., n

Now p $\in$ G_1, and G_1 is open.

$\therefore$ p is an interior point of G_1

$\therefore$ There exists $\delta_1 > 0$ such that

$$]\, p-\delta_1,\ p+\delta_1\, [\leq G_1$$

Similarly, p is an interior point of G_2.

$\therefore$ There exists $\delta_2 > 0$ such that

$$]\, p-\delta_2,\ p+\delta_2\, [\subseteq G_2$$

In this way, p is an interior point of G_n.

$\therefore$ There exists $\delta_n > 0$ such that

$$]\, p-\delta_n,\ p+\delta_n\, [\subseteq G_n$$

Let $\delta = \min\ \{\delta_1, \delta_2, ..., \delta_n\}$

Then $\delta > 0$

Now $]p-\delta,\ p+\delta[\subseteq]p-\delta_1, p+\delta_1[\subseteq G_1$

$]p-\delta,\ p+\delta[\subseteq]p-\delta_2, p+\delta_2[\subseteq G_2$

$]p-\delta,\ p+\delta[\subseteq]p-\delta_n, p+\delta_n[\subseteq G_n$

$\therefore$ $]p-\delta,\ p+\delta[\leq G_1 \cap G_2 \cap .... \cap G_n = G$

$\therefore$ p is an interior point of G.

$\therefore$ G is an open set.

Example 1 : Examine if the intersection of the family of open sets $G_n =]-\frac{1}{n}, \frac{1}{n}[$, $n \in N$ is open.

Solution : We know that, for each $n \in N$, $]-\frac{1}{n}, \frac{1}{n}[$ is open

Now, $G_1 =]-1, 1\ [$

$G_2 =]-\frac{1}{2}, \frac{1}{2}[$

$G_3 =]-\frac{1}{3}, \frac{1}{3}[$

$G_n =]-\frac{1}{n}, \frac{1}{n}[$

Each G_i is open.

$\bigcap_{i=1}^{\infty} G_i = \{0\}$, which is not open.

Example 2 : Examine if the intersection of the family of open sets

$G_n =]-n, n[$, $n \in \underline{\underline{N}}$ is open.

Solution: We know that, for each $n \in \underline{\underline{N}}$,

] – n, n [is open.

Now $G_1 =]-1, 1[$

$G_2 =]-2, 2[$

$G_3 =]-3, 3[$

.....

.....

$G_n =]-n, n[$

.....

Each G_i is open

$\bigcap_{i=1}^{\infty} Gi =]-1, 1[= G_1$, which is open.

Theorem 2.13: Let S be a subset of R. Then int S is an open set.

Proof: If int S = ϕ, there is nothing to prove.

ϕ is an open set.

We assume that int $S \neq \phi$

Let $p \in$ int S

$\therefore$ p is an interior point of S.

$\therefore$ There exists a nbd N(p) of p such that $N(p) \subseteq S$.

Let q $\in$ N(p)

$\therefore$ N(p) is also a nbd of q.

$\therefore$ $N(p) \subseteq S$, q is an interior point of S.

Thus q $\in$ N(p) $\Rightarrow$ q $\in$ int S

$\therefore$ $N(p) \subseteq$ int S

$\therefore$ p is an interior point of int S

$\therefore$ int S is an open set.

Theorem 2.14 : The interior of a set S is the largest open subset of S.

Proof: We know that int S is open. Again, by definition, int $S \subseteq S$.

We are to prove that int S is the largest open subset of S.

Let T be any open subset of S.

Let $P \in T$

$\therefore$ T is open, p is an interior point of T.

$\therefore$ There exists a nbd N(p) of p such that $N(p) \subseteq T$.

But $T \subseteq S$

$\therefore$ $N(p) \subseteq S$.

$\therefore$ p is an interior point of S.

$\therefore$ $p \in$ int S

Thus $p \in T \Rightarrow p \in$ int S

$\therefore$ $T \subseteq$ int S.

Thus, the interior of S is the largest open subset of S, i.e., the interior of S contains every open subset of S.

Corollary

We have seen that the interior of S contains every open subset of S.

$\therefore$ int S = union of all open subsets of S.

Let Us Recall

The difference of two sets A and B, denoted by A – B, is the set of elements which belong to A, but which do not belong to B, i.e.,

A– B = $\{x : x \in A, x \notin B\}$

2.8 CLOSED SET

Definition 1 : Let S be a subset of R. S is said to be closed if R – S is open.

Examples :

We know that R is an open set. ϕ is the complement of R. Therefore, ϕ is a closed set.

Again, ϕ is an open set. R is the complement of ϕ. Therefore, R is a closed set.

Note:

1. R is both open and closed.
2. ϕ is both open and closed.

Theorem 2.14: Let S be a subset of R. S is closed if and only if $S' \leq S$, where S' denotes the derived set of S.

Proof: Let S be closed

$\therefore$ R – S is open.

Let $p \in R - S$

$\therefore$ p is an interior of $R - S.$

$\therefore$ There exists a nbd N(p) of p such that N(p) $\leq R - S.$

$\Rightarrow N(p) \cap S = \phi$

$\therefore$ p is not a limit point of S.

$\therefore p \notin S'$

Thus, $p \in R - S \Rightarrow p \notin S'$

This implies $p \in S' \Rightarrow p \notin R - S$, i.e., $p \in S$

Thus, $p \in S' \Rightarrow p \in S$

$\therefore S' \subseteq S$

Conversely, let S be a subset of R such that $S' \leq S$.

Let $p \in R - S$

$\Rightarrow p \notin S$

$\Rightarrow p \notin S', \quad (\because S' \subseteq S)$

$\therefore$ There exists a nbd N(p) of p such that N(p) $\cap S = \phi$

$\therefore$ N(p) $\subseteq R - S$

Thus, $p \in R - S \Rightarrow N(p) \subseteq R - S$

$\therefore$ $R - S$ is open

$\therefore$ S is closed.

In view of the above theorem, a closed set can also be defined as follows:

Definition 2 : Let S be a subset of R S is said to be closed if $S' \subseteq S$, where S' denotes the derived set of S.

Note: The definitions 1 and 2 are equivalent.

Remark: Note the following two questions.

Q 1. Define a closed set. Prove that a subset S of R is closed if and only

if $S' \leq S$.

Q 2. Define a closed set. Prove that a subset S of R is closed if and only if R – S is open.

For Q 1, you will have to write definition 1, and prove theorem 2.14

Fr Q 2, you will have to write definition 2, and prove theorem 2.14.

Example 1 : Let us consider the set N of natural numbers.

We know that N has no limit point.

$\therefore N' = \phi$. So $N' \subseteq N$

$\therefore$ N is a closed set.

2. Let us consider the set Z of integers.

We know that Z has no limit point.

$\therefore Z' = \phi$. So $Z' \subseteq Z$

$\therefore$ Z is a closed set.

3. Let us consider the set Q of rationals. We know that $Q' = R$. So $Q' \subseteq Q$

$\therefore$ Q is not closed.

Again every nbd of $p \in Q$ contains rationals and irrationals. Therefore no point of Q is an interior point of Q.

$\therefore$ Q is not open.

Note: Q is neither open nor closed.

Theorem 2.15 : The union of two closed sets in R is closed.

Proof : Let F_1, F_2 be two closed sets of R.

$\therefore F_1' \subseteq F_1, \ F_2' \subseteq F_2$

We have $(F_1 \cup F_2)' = F_1' \cup F_2'$ (thm 2.3 (b))

Now $F_1' \subseteq F_1 \subseteq F_1 \cup F_2$

$F_2' \subseteq F_2 \subseteq F_1 \cup F_2$

$\therefore F_1' \cup F_2' \subseteq F_1 \cup F_2$

$\Rightarrow (F_1 \cup F_2)' \subseteq F_1 \cup F_2$

$\Rightarrow F_1 \cup F_2$

Theorem 2.16: The union of a finite number of closed sets in R is closed.

Proof: Let F_1, F_2, ..., F_n be n closed sets in R. Let $F = F_1 \cup F_2 \cup .... \cup F_n$

Now $F_1' \subseteq F_1, F_2' \subseteq F_2, \ldots, F_n' \subseteq F_n$

We have $(F_1 \cup F_1 \cup \ldots \cup F_n)' = F_1' \cup F_2' \cup \ldots \cup F_n'$ i.e.

$F' = F_1' \cup F_2' \cup \ldots \cup F_n'$

Now $F_1' \subseteq F_1 \subseteq F$

$F_2' \subseteq F_2 \subseteq F$

.....

.....

$F_n' \subseteq F_n \subseteq F$

$\therefore F_1' \cup F_2' \cup \ldots \cup F_n' \subseteq F$

$\Rightarrow F' \subseteq F$

$\therefore F = F_1 \cup F_2 \cup \ldots \cup F_n$ is closed.

Example 3 : Examine if the union of the family of closed sets

$F_n = \left[-\frac{1}{n}, \frac{1}{n}\right], n \in N$ is closed.

Solution: Each point of $F_n, n \in N$ is a limit point of F_n. $\therefore F_n' \subseteq F_n$

$\therefore F_n$ is closed, $n \in N$.

N ow $F_1 = [-1, 1]$

$F_2 = \left[-\frac{1}{2}, \frac{1}{2}\right]$

$F_3 = \left[-\frac{1}{3}, \frac{1}{3}\right]$

.....

.....

$F_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$

Now $F_1 \cup F_2 \cup F_3 \cup \ldots \cup F_n \cup \ldots$

$= \bigcup_{i=1}^{\infty} F_i = [-1, 1] = F_1$, which is closed.

Example 4: Examine if the union of the family of closed sets

$F_n = \left[\frac{1}{n}, 4 - \frac{1}{n}\right], n \in N$ is closed.

Solution: Each point of F_n'. $\therefore F_n' \subseteq F_n$. $\therefore F_n$ is closed.

Now $F_1 = [1, 3]$

$$F_2 = \left[\frac{1}{2}, 4-\frac{1}{2}\right]$$

$$F_3 = \left[\frac{1}{3}, 4-\frac{1}{3}\right]$$

.....

.....

$$F_n = \left[\frac{1}{n}, 4-\frac{1}{n}\right]$$

Now $F_1 \cup F_2 \cup F_3 \cup .... \cup F_n \cup ....$

$= \bigcup_{i=1}^{\infty} F_i =]0, 3[$, which is not closed.

Each point of] 0, 3 [is a limit point of] 0, 3 [. Moreover, 0, 3 are limit points of] 0, 3 [. But 0, 3 do not belong to] 0, 3 [. So,]0, 3 [is not closed.

Do You Know ?

Any closed interval of the real numbers is a closed set.

Let *a, b* be reals, *a* < *b*.

[*a, b*] is a closed set.

] *a, b* [is not closed.

Theorem 2.17 : The intersection of two closed sets in R is closed.

Proof: Let F_1, F_2 be two closed sets in R. Let $F = F_1 \cap F_2$

Now, $F_1' \subseteq F_1, F_2' \subseteq F_2$

$F' = (F_1 \cap F_2)' \subseteq F_1' \cap F_2'$ (Thm 2.3 (c))

Now $F' \subseteq F_1' \cap F_2' \subseteq F_1' \subseteq F_1$

$F' \subseteq F_1' \cap F_2' \subseteq F_2' \subseteq F_2$

$\therefore\ F' \subseteq F_1 \cap F_2$, i.e., $F' \subseteq F$

$\therefore\ F = F_1 \cap F_2$

Theorem 2.18 : The intersection of a finite number of closed sets in R is closed.

Proof : Let F_1, F_2, ..., Fn be n closed sets in R. Let $F = F_1 \cap F_2 \cap ..... \cap F_n$

Now, $F_1' \subseteq F_1,\ F_2' \subseteq F_2, ..., F_n' \subseteq F_n$

$F' = (F_1 \cap F_2 \cap ..... \cap F_n)' \subseteq F_1' \cap F_2' \cap ..... \cap F_n'$

Now, $F' \subseteq F_1' \cap F_2' \cap ..... \cap F_n' \subseteq F_1$

Similarly, $F' \subseteq F_2, ..., F' \subseteq F_n$

$\therefore\ F' \subseteq F_1 \cap F_2 \cap ..... \cap F_n \subseteq F_1$, i.e., $F' \subseteq F$

$\therefore$ F is closed.

Theorem 2.19: The intersection of an arbitrary collection of closed sets in $\underline{\underline{R}}$ is closed.

Proof: Let $\{F_i : i \in I\}$ be an arbitray collection of closed sets.

Let $F = \bigcap_{i \in I} F_i$

If $F' = \phi$, there is nothing to prove.

Clearly $F' \subseteq F$, and so F is closed.

We assume that $F' \neq \phi$.

Let $p \in F'$.

$\therefore$ p is a limit point of F.

We choose $\delta > 0$

Then $]p - \delta, p + \delta[$ contains a point, say q, of F.

$q \in]p - \delta, p + \delta[\cap F$

$\Rightarrow q \in]p - \delta, p + \delta[\cap F_j$, for each $j \in I$

$\therefore$ p is a limit point of F_j, for each $j \in I$

Hence $p \in \bigcap_{i \in I} F_i$, i.e., $p \in F$

Thus, $p \in F' \Rightarrow p \in F$

$\therefore\ F' \subseteq F$

$\therefore$ F is closed.

2.9 CLOSURE OF A SET

Let S be a subset of the set of real numbers. A point $p \in \underline{\underline{R}}$ is said to be an adherent point of S if every nbd of p contains at least one point of S.

Note 1 : Clearly a point $p \in S$ is an adherent point of S. Every nbd of S contains at least one point of S, viz., the point itself.

2. If $p \notin S$, then p is an adherent point of S only if p is a limit point of S.

3. If An adherent point may or may not belong to the set.

4. An adherent point may or may not be a limit point.

Definition: The set of all adherent points of S is called the closure of S. Generally it is denoted by $\bar{S}$.

Theorem 2.20 : Let S be a subset of R. Then $\bar{S} = S \cup S'$, where S' denotes the derived set of S.

Proof : Let $p \in S$

Then every nbd of p contains p.

$\therefore$ p is an adherent point of S.

$\therefore\ p \in \bar{S}$

Thus $p \in S \Rightarrow p \in \bar{S}$

$\therefore\ S \subseteq \bar{S}$ (1)

Again, let $q \in S'$

$\therefore$ q is a limit point of S.

$\therefore$ every nbd of q contains a point of S other than q.

$\therefore\ q \in \bar{S}$

Thus, $q \in S' \Rightarrow q \in \bar{S}$

$\therefore\ S' \subseteq \bar{S}$ (2)

From (1) and (2), $S \cup S' \subseteq S'$ (3)

Again let $r \notin SUS'$

$\Rightarrow r \notin S$ and $\Rightarrow r \notin S'$

$\therefore\ r \notin S'$, there exists a nbd N(r) of r such that $N'(r) \cap S = \phi$

Again, since $r \notin S,\ N'(r) \cap S = \phi \Rightarrow N(r) \cap S = \phi$

$\therefore$ r is not an adherent point of S.

$\therefore\ r \notin \bar{S}$

Thus $r \notin S \cap S' \Rightarrow r \notin \bar{S}$

$\therefore\ \bar{S} \subseteq S \cup S'$ (4)

From (3) and (4)

$S \cup S' = \bar{S}$

Do You Know ?

The contrapositive of the statement “if p, then q” is “if not q, then not p”.

A statement and its contrapositive are logically equivalent.

Let A, B be two sets

If $x \in A \Rightarrow x \in B,$ then $A \subseteq B$ contrapositively,

if $x \notin B \Rightarrow x \notin A,$ then $A \subseteq B$

Theorem 2.21: Let S be a subset of R. Then S is closed if and only if $S = \bar{S}$.

Proof : Let S be a closed set.

$\therefore\ S' \subseteq S$

$\therefore\ S \cup S' = S$

$\Rightarrow \bar{S} = S$ (Thm 2.20)

Conversely, let S be a subset of $\underline{\underline{R}}$ such that $S = \bar{S}$.

This $\Rightarrow S = S \cup S'$

$\Rightarrow S' \leq S$

$\Rightarrow$ S is closed.

Example 5 : Find the closure of the set $S = \left\{\frac{1}{n} : n \in N\right\}$

Solution: We have $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$

We know that 0 is the only limit point of S.

$\therefore\ S' = (0)$

Thus, $\bar{S} = S \cup S'$

$$= \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\} \cup \{0\}$$

$$= \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$$

Example 6 : Find the closures of

(i) N (ii) Z (iii) Q (iv) R

Solution: (i) Let $p \in R$

No nbd of p contains of point of N other than p. $\therefore\ N' = \phi$.

$\therefore\ \bar{N} = N \cup N' = N \cup \phi = N$

(ii) We have $Z' = \phi$

$\therefore \bar{Z} = Z \cup Z' = Z \cup \phi = Z$

(iii) We have $\underline{\underline{Q'}} = \underline{\underline{R}}$

$\therefore\ \bar{Q} = Q \cup Q' = Q \cup R = R$

(iv) We have $\underline{\underline{R'}} = \underline{\underline{R}}$

$\therefore\ \bar{R} = R \cup R' = R \cup R = R$



CHECK YOUR PROGRESS 3

Q 1: Show that the set $\left\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \ldots.\right\}$ is neither open nor closed.

Q 2: Show that the set

$S = \left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots.\right\}$ is closed but not open.

Q 3: Examine if the union ofo the family of closed sets

$F_n = \left[\frac{1}{n}, 2\right], n \in N$ is closed.

2.10 COUNTABILITY OF SETS

Definition 1 : Let S be a subset of R. S is said to be denumerable if there exists a bijective mapping $f : N \to S$. A denumerable set is also called countably infinite set.

Definition 2 : A set which is either finite or denumerable is said to be countable.

A set, which is not countable, is said to be uncountable.

Examples :

1. Let us consider the set N of natural numbers.

   Let $f : N \to N, f(n) = n$

   Let $x_1, x_2 \in N$

   Then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

   $\therefore$ f is one-one.

   For every y of N, there exists $x \in N$ such that $f(x) = y$

   $\therefore$ f is onto

   $\therefore$ f is a bijective mapping

   $\therefore$ N is countable.

2. Let us consider the set Z of integers.

   Let $f : Z \to N,$

$$f(x)=\begin{cases}\frac{1}{2}n, \text{if } n \text{ is even}\\ \frac{1}{2}(1-n), \text{if } n \text{ is odd.}\end{cases}$$

Clearly f is one-one and onto, i.e., f is a bijective mapping.

$\therefore$ Z is countable

Example 7 : Show that the set E of even natural numbers is countable.

Solution: Let $f:\mathrm{N}\to\mathrm{E},\ f(n)=2n$

Clearly f is one-one and onto.

$\therefore$ E is countable.

Note: If S is denumerable, there exists a bijective mapping f from N to S. Thus, corresponding to every element $n\in\mathrm{N}$, there exists $f(n)\in\mathrm{S}$ such that the elements of S can be described as $f(1), f(2), ..., f(n), ...$ or as $a_1, a_2, ..., a_n, ...$

Theorem 2.22: Every subset of a countable set is countable.

Proof : Let S be countable. Then S is either finite or denumerable.

Let A be a subset of S.

Case 1. Let S be finite.

Then A is also finite, and so it is countable.

Case 2. Let S be denumerable, and A be finite.

Clearly A is countable.

Case 3. Let S be denumerable, and A be infinite.

Now S can be written as

$\mathrm{S}=\{a_1, a_2, a_3, ...\}$

Let n_1 be the smallest subscript for which $a_{n_1}\in\mathrm{A}$.

We consider a new set $\mathrm{A}-\{a_{n_1}\}$.

Let n_2 be the smallest subscript such that $a_{n_2}\in\mathrm{A}-\{a_{n_1}\}$.

In this way, let n_k be the smallest subscript such that

$a_{n_k}\in\mathrm{A}-\{a_1, a_2, ..., a_{n_{k-1}}\}$

Since A is infinite, such an element a_{n_k} always exists for each $k\in\mathrm{N}$

Now A – $\{a_{n_1}, a_{n_2}, ..., a_{n_k}\}\neq\phi$, for each $k\in\mathrm{N}$

So we can write

A = $\{a_{n_1}, a_{n_2}, ..., a_{n_k}, ...\}$

We define a mapping

$f : \mathrm{N} \to \mathrm{A},\ f(k) = a_{n_k}$

Clearly f is one-one and onto.

$\therefore$ A is denumerable, and so, it is countable.

Theorem 2.23 : The union of two countable sets is countable.

Proof : Let S and T be two countable sets, where $\mathrm{S} \cap \mathrm{T} = \phi$

Case 1 : Let S and T be finite. Then $\mathrm{S} \cup \mathrm{T}$ is also finite, and so, it is countable.

Case 2 : Let S be finite and T be denumerable.

Let S = $\{a_1, a_2, ..., a_k\}$

T = $\{b_1, b_2, ..., b_n, ...\}$

We define $f : \mathrm{S} \cup \mathrm{T} \to \mathrm{N},$ by the rule

$f(a_i) = i,$ for $1 \le i \le k$

$f(b_n) = n + k,$ for each n

Clearly f is one-one and onto.

$\therefore$ $\mathrm{S} \cup \mathrm{T}$ is denumerable, and, so countable.

Case 3 : Let S and T be denumerable.

Let S = $\{a_1, a_2, a_3, ..., a_n, ...\}$

T = $\{b_1, b_2, b_3, ..., b_n, ...\}$

We define $\mathrm{S} \cup \mathrm{T} \to \mathrm{N},$ by the rule

$f(a_n) = 2n$

$f(b_n) = 2n - 1$

Clearly f is one-one and onto, and, so countable.

Note: 1. The Theorem holds good if $\mathrm{S} \cap \mathrm{T} \neq \phi$.

2. The union of a countable number of countable sets is countable.

Example 8 : Show that the set of all rational numbers between [0, 1] is countable.

Solution: Let us construct the following sets.

$\mathrm{A}_1 = \{0, 1\}$

$\mathrm{A}_2 = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\right\}$

$\mathrm{A}_3 = \left\{\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, ...\right\}$

$$A_4 = \left\{\frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, ...\right\}$$

$$A_5 = \left\{\frac{4}{5}, \frac{4}{7}, \frac{4}{9}, \frac{4}{11}, ...\right\}$$

.....

.....

We note that each of the above sets is countable.

$$\therefore \cup A_i = A_1 \cup A_2 \cup A_3 \cup A_4 \cup ....$$

$$= \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, ...\right\}$$

= [0, 1]

Which is countable.

Example 9 : Prove that the set Q of all rational numbers is countable.

Proof Case 1 : First we consider the set Q_+ of all positive rational numbers.

Let us construct the following sets.

$$A_1 = \{1, 2, 3, ...\}$$

$$A_2 = \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...\right\}$$

$$A_3 = \left\{\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, ...\right\}$$

$$A_4 = \left\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, ...\right\}$$

$$A_5 = \left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{6}{5}, \frac{7}{5}, ...\right\}$$

.....

.....

We note that each of the above sets is countable. Now we list the elements as follows:

$$A_1 : 1,\ 2,\ 3,\ 4,\ ....$$

$$A_2 : \frac{1}{2},\ \frac{2}{2},\ \frac{3}{2},\ \frac{4}{2},\ ....$$

$$A_3 : \frac{1}{3},\ \frac{2}{3},\ \frac{3}{3},\ \frac{4}{3},\ ....$$

$$A_4 : \frac{1}{4},\ \frac{2}{4},\ \frac{3}{4},\ \frac{4}{4},\ ....$$

$A_5 : \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, ...$

.....

.....

Thus, we get

$$\underline{\underline{Q}}_{+} = \left\{1, \frac{1}{2}, 2, \frac{1}{3}, \frac{2}{3}, 3, \frac{1}{4}, \frac{3}{4}, \frac{4}{3}, ...\right\}$$

$$= \cup A_i, (i = 1, 2, 3, ...)$$

which is countable.

Case 2 : Next we consider the set Q_ of all negative rational numbers.

We define a mapping $f : Q_+ \to Q_-$ by the rule $f(x) = -x, \forall x \in Q_+$

Clearly f is one-one and onto.

Since Q_+ is countable, therefore Q_ is also countable.

Case 3 : Clearly {0} is countable.

$\therefore\ Q = Q_- \cup \{0\} \cup Q_+$ is countable.

Theorem 2.24 : The open interval] 0, 1 [is not countable.

Proof : Let I =] 0, 1 [

$$= \{x \in R : 0 < x < 1\}$$

If possible, let I be denumerable. Then the elements of I can be described as $a_1, a_2, a_3, ...$

We know that every rational number can be expressed as a terminating decimal or as a non-terminating but recurring decimal. On the other hand every irrational number is a non-terminating and non-recurring decimal.

Now, a terminating decimal can be expressed as a non-terminating decimal either by using 9's or by using 0's. For example 0.15 can be written as 0.14999.... or 0.15000.....

We use only 9's and ignore 0's. Then every real number in I will have a unique non-terminating decimal representation.

Let $a_1 = 0.a_{11}\, a_{12}\ a_{13}....$

$a_2 = 0.a_{21}\, a_{22}\ a_{23}....$

$a_3 = 0.a_{31}\, a_{32}\ a_{33}....$

.....

..... where $0 < a_{ij} < 9$.

We have assumed that every real number in I has a place in the enumeration $a_1, a_2, a_3, ...$

We consider the real number

$$x = 0.x_1x_2x_3....,$$

where $x_k \begin{cases} = 1, \text{ if } a_{kk} \neq 1 \\ = 2, \text{ if } a_{kk} = 1 \end{cases}$

Since *x* differs from a_i in the ith decimal place, therefore *x* is different from each a_i.

$\therefore \quad x \notin \{a_1, a_2, a_3, ...\}$

But 0 < *x* < 1, i.e., $x \in \mathrm{I}$

Now $x \in \mathrm{I}$, but $x \notin \{a_1, a_2, a_3, ...\}$

This is a contradiction.

So, our assumption is wrong.

$\therefore$ I is not denumerable, and, so, not countable.

Note: We know that if a set is countable, every subset of it is also countable. So, if a set is not countable, its superset is also not countable.

Since I =] 0, 1 [is not countable, therefore R is also not countable.



2.11 LET US SUM UP

- A subset S of R is said to be a neighbourhood (nbd) of a real number p if there exists an open interval] *a, b* [such that $\mathrm{p} \in]\mathrm{a}, \mathrm{b}[\subseteq \mathrm{S}$
- Let S be a subset of R. A real number p is said to be a limit point of S if every nbd of p contains a point of S other than p.
- The set of all points of S is called the derived set of S. Generally it is denoted by S'.
- Every infinite bounded subset of R has a limit point of R. This result is known as the Bolzano-Weierstrass Theorem.
- Let S be a subset of R. A point $\mathrm{p} \in \mathrm{S}$ is said to be an interior point of S if there exists a nbd N(p) of p such that $\mathrm{N(p)} \subseteq \mathrm{S}$.
- The set of all interior points of S is called the interior of S. Generally it is denoted by int S.

- Let S be a subset of R. S is said to be an open set if every point of S is an interior point of S.
- The interior of a set S is the largest open subset of S.
- Let S be a subset of R. S is said to be closed if R – S is open.
- Let S be a subset of R. S is closed if and only if $S' \subseteq S$, where S' denotes the derived set of S.
- Let S be a subset of R. S is said to be closed if $S' \subseteq S$.
- R and ϕ are both open and closed.
- Q is neither open nor closed.
- Let S be a subset of R. A point $p \in R$ is said to be an adherent point of S if every nbd of p contains at least one point of S.
- The set of all adherent points of S is called the closure of S.
- A set which is either empty, finite or countably infinite is called a countable set.



2.12 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: Given that S is a nbd of a real number p.

$\therefore$ There exists an open interval of the form $]p-\delta, p+\delta[$ for same $\delta > 0$ such that

$]p-\delta, p+\delta[\subseteq S$ (1)

Again $S \subseteq T$, (2) (given)

From (1) and (2) $]p-\delta, p+\delta[\subseteq T$

$\therefore$ T is also a nbd of p.

Ans to Q No 2: Let $p \in R-S$

$\therefore p \notin S$

If possible, let S be a nbd of p.

$\therefore$ There exists an interval $]p-\delta, p+\delta[$ for same $\delta > 0$ such that

$]p-\delta, p+\delta[\subseteq S$

$\Rightarrow p \in S,$ which is a contradiction.

$\therefore$ S cannot be a nbd of any point of R – S.

Ans to Q No 3:

(a) The set of real number R is a nbd of each real number p, for any open interval containing ps is a subset of R.

(b) The set of rational numbers Q is not a nbd of any of its points p, for any open interval containing p will contain infinite number of irrational numbers. Clearly the open interval cannot be a subset of Q.

Check Your Progress 2

Ans to Q No 1: Let $p \in N$.

Every nbd of p contains points not belonging to N. So, p cannot be an interior point of N. So, int $N = \phi$.

No point of N is an interior point of N. So N is not an open set.

Ans to Q No 2: int $Z = \phi$

Z is not an open set.

Ans to Q No 3: int $Q = \phi$

Q is not an open set.

Ans to Q No 4: Every point of R is an interior point of R

R is an open set.

Ans to Q No 5:

(i) Let us consider] *a, b* [

Let $p \in]a,b[$

$\therefore a < p < b$

Let c, d be two real numbers such that

$a < c < p, \quad p < d < b$

i.e., $a < c < p < d < b$

i.e., $p \in]c,d[\subseteq]a,b[$

Thus, p is an interior point of] *a, b* [i.e., every point of] *a, b* [is an interior point of] *a,b*[

$\therefore$] *a, b* [is an open set.

(ii) Let us consider] *a, b* [

a is not an interior point of [*a, b* [

$\therefore$] *a, b* [is not an open set.

(iii) Let us consider] *a, b*]

b is not an interior point of] *a, b*]

$\therefore$] *a, b*] is not an open set.

(iv) Let us consider [*a, b*]

a, b are not interior points of [*a, b*]

$\therefore$ [*a, b*] is not an open set.

Ans to Q No 6: Let S be a nonempty finite set.

Let $S = \{x_1, x_2, ..., x_n\}$

Let $p \in S$

If p is an interior point of S, then there exists a nbd N(p) of p such that N(p) $\subseteq$ S. This is not possible, since N(p) contains infinite number of elements, whereas S is finite.

Thus, no point of S is an interior point of S.

$\therefore$ S is not an open set.

Check Your Progress 3

Ans to Q No 1: We have $S = \left\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, ...\right\}$

Every nbd of 0 contains an infinite number of points of S.

$\therefore$ 0 is a limit point.

But $0 \notin S$. $\therefore$ S is not closed.

Let $p \in S$

Every nbd of p contains points not belonging to S.

$\therefore$ no point of S is an interior point of S.

$\therefore$ S is not open.

Ans to Q No 2: –1, and 1 are the only interior points of the set. No other elements of S are interior points. Therefore, S is not open.

–1, 1 are the only limit points of the set, and $-1 \in S, 1 \in S$. $\therefore$ S is closed.

Ans to Q No 3: Each point of F_n, $n \in N$ is a limit point of F_n. $\therefore F_n' \subseteq F_n$

$\therefore$ F_n is closed, $n \in N$

Now, $F_1 = [1, 2]$

$F_2 = \left[\frac{1}{2}, 2\right]$

$F_3 = \left[\frac{1}{3}, 2\right]$

$F_4 = \left[\frac{1}{4}, 2\right]$

.....

.....

$$F_n = \left[\frac{1}{n}, 2\right]$$

Now $F_1 \cup F_2 \cup F_3 \cup ..... \cup F_n \cup ....$

$= \bigcup_{i=1}^{\infty} F_i =]0, 2],$ which is not closed.



2.13 FURTHER READING

1) Principles of Mathematical Analysis by Walter Rudin.
2) Methods of real Analysis by Richand R. Goldbery.
3) Mathematical Analysis by S. C. Malik and Savita Arora.
4) A Course of Mathematical Analysis by shanti Narayan.



2.14 MODEL QUESTIONS

Q.1: Define neighbourhood of a point. Is the arbitrary intersection of the neighbourhoods of a point a neighbourhood of the point? Justify your answer.

Q.2: Show that the set of real numbers is the derived set of the set of rational numbers.

Q.3: Prove that every open set is a union of open intervals.

Q.4: Is every infinite set open? Justify your answer.

Q.5: Let G be an open subset of R, and F a closed subset of R.
Prove that
(i) G – F is open,
(ii) F – G is closed.

Q.6: State and prove Bolzano-Weierstrass Theorem. Verify the theorem for the set

$$S = \left\{\frac{n-1}{n+1} : n \in N\right\}$$

Q.7: Show that a finite set has no limit points.

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